Bias of Least Squares Approaches for Diffusion Tensor Estimation from Array Coils in DT–MRI

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Abstract

Least Squares (LS) and its weighted version are standard techniques to estimate the Diffusion Tensor (DT) from Diffusion Weighted Images (DWI). They require to linearize the problem by computing the logarithm of the DWI. For the single-coil Rician noise model it has been shown that this model does not introduce a significant bias, but for multiple array coils and parallel imaging, the noise cannot longer be modeled as Rician. As a result the validity of LS approaches is not assured. An analytical study of noise statistics for a multiple coil system is carried out, together with the Weighted LS formulation and noise analysis for this model. Results show that the bias in the computation of the components of the DT may be comparable to their variance in many cases, stressing the importance of unbiased filtering previous to DT estimation.

Statistics of noise in the log–domain

Assuming the signal A_i real, the received signal M_i is described

$$M_i = \sqrt{\left(A_i + n_{c,i}^2\right)^2 + n_{s,i}^2} \qquad M_i = \sqrt{\sum_{l=1}^{L}\left(A_{i,l} + n_{c,i,l}^2\right)^2 + n_{s,i,l}^2}$$

Single coil **(Rician)** Multiple coil **(Non–central Chi)**

Statistics related to the logarithm of M

$$E\{\log(M)\} = \frac{1}{2}\log\left(2\sigma_n^2\right) + \frac{1}{2}\frac{a}{L}{}_2F_2(1,1:2,1+L;-a) + \frac{1}{2}\psi(L);$$

$$\mathrm{Var}\{\log(M)\} = \frac{1}{4}\left[\tilde{N}_L(a) - 2\log(2\sigma^2)\frac{a}{L}{}_2F_2(1,1:2,1+L;-a) - \left(\psi(L) + \frac{a}{L}{}_2F_2(1,1:2,1+L;-a)\right)^2\right];$$

After simplification, bias and variance:

$$\left.\begin{aligned}\mathrm{Var}\{\log(M)\} &= \tfrac{1}{2}a^{-1} - \tfrac{3L-4}{4}a^{-2} + \mathscr{O}\left(a^{-3}\right)\\ \mathrm{bias}^2\{\log(M)\} &= \tfrac{(L-1)^2}{4}a^{-2} + \mathscr{O}\left(a^{-3}\right)\end{aligned}\right\} \text{ with } a = \frac{A_L^2}{2\sigma^2} = \frac{\mathrm{SNR}^2}{2}$$

Tensor fitting based on Weighted Least Squares

Estimation of the DT coefficients as a WLS problem:

$$\log(A_0) - \log(M_i) = \left[g_{i,1}^2,\ 2g_{i,1}g_{i,2},\ 2g_{i,1}g_{i,3},\ g_{i,2}^2,\ 2g_{i,2}g_{i,3},\ g_{i,3}^2\right] \cdot \left[bD_{11},\ bD_{12},\ bD_{13}, bD_{22},\ bD_{23},\ bD_{33}\right]^T + \varepsilon_i$$

$$G^TW(\mathbf{Y} - G\mathbf{X}) = 0 \Rightarrow \mathbf{X} = \left(G^TWG\right)^{-1}G^TW\mathbf{Y}$$

with $W = \mathrm{diag}(W_{ii})$. We will fix:

$$W_{ii} = \mathrm{Var}^{-1}\{\mathbf{Y}_i\} \simeq \frac{1}{a_i^{-1}/2 - a_i^{-2}(3L-4)/4} \simeq 2a_i + (3L-4) \to 2a_i$$

1. Formulation identical to the traditional WLS for Rician noise.
2. The weights W_{ii} are proportional to $2a_i \Rightarrow$ not necessary to know the value of σ.

Variance and bias on the tensor components

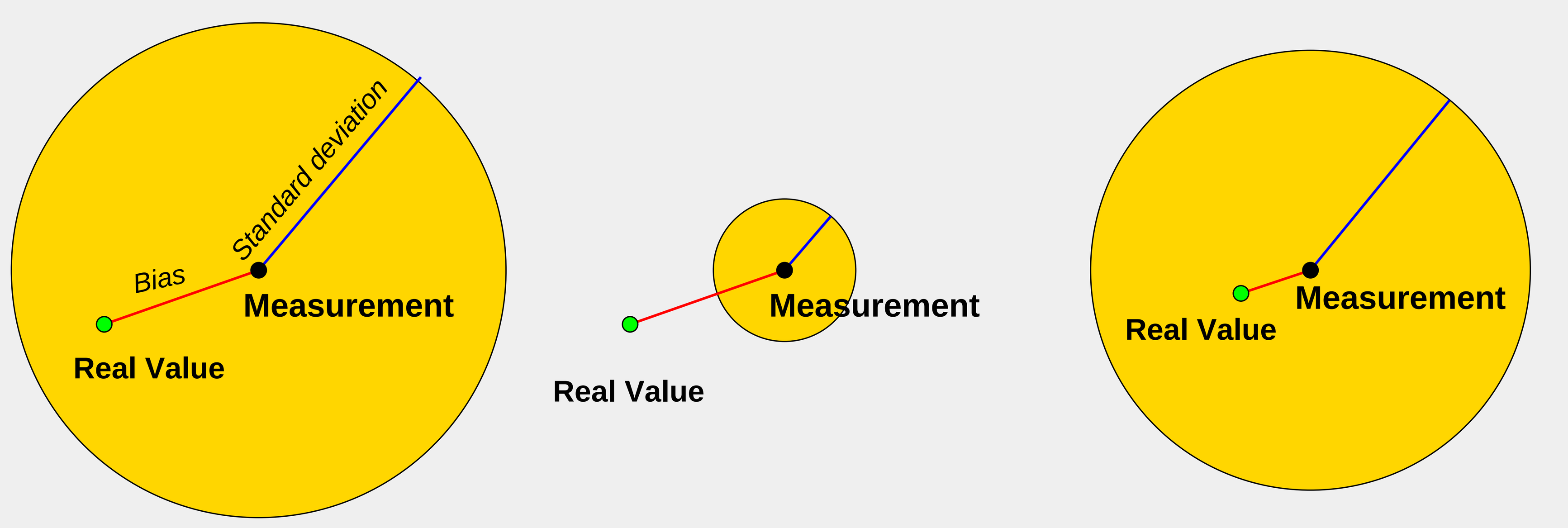
Variance and **bias** using the simplifications:

$$\left.\begin{aligned}C_{\mathbf{XX}} &= E\left\{(\mathbf{X} - E\{\mathbf{X}\})(\mathbf{X} - E\{\mathbf{X}\})^T\right\}\\ &= \mathscr{L}G^TWC_{\mathbf{YY}}W^TG\mathscr{L}^T\\ &\simeq \mathscr{L} - (3L-4)\mathscr{L}G^TG\mathscr{L}\end{aligned}\right\} \text{ with } \mathscr{L} = (G^TWG)^{-1}$$

$$\mathrm{bias}\{\mathbf{X}\} = \mathscr{L}G^TW\mathrm{bias}\{\mathbf{Y}\} \simeq (L-1)\mathscr{L}G^T\mathbf{e}_1\} \text{ with } \mathbf{e}_1 \text{ vector of 1's}$$

The error (**MSE**) is defined as

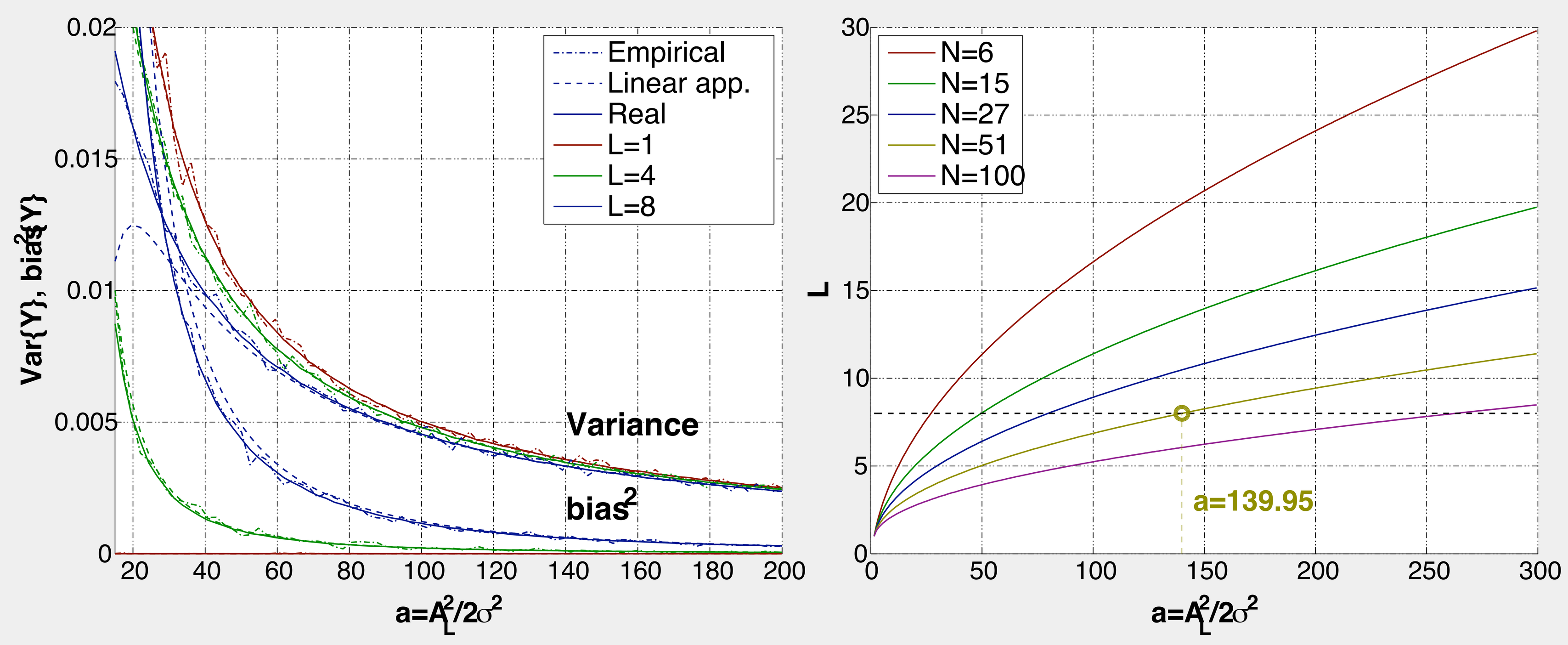
$$\mathrm{MSE} = \mathrm{Var}\{\mathbf{X}\} + \mathrm{bias}^2\{\mathbf{X}\} = b^2\left(\Delta_{11}^2 + \Delta_{12}^2 + \Delta_{13}^2 + \Delta_{22}^2 + \Delta_{23}^2 + \Delta_{33}^2\right)$$
$$\simeq \mathrm{tr}(\mathscr{L}) - (3L-4)\mathrm{tr}\left(\mathscr{L}G^TG\mathscr{L}\right) + (L-1)^2N^2v^T\mathscr{L}^2v$$



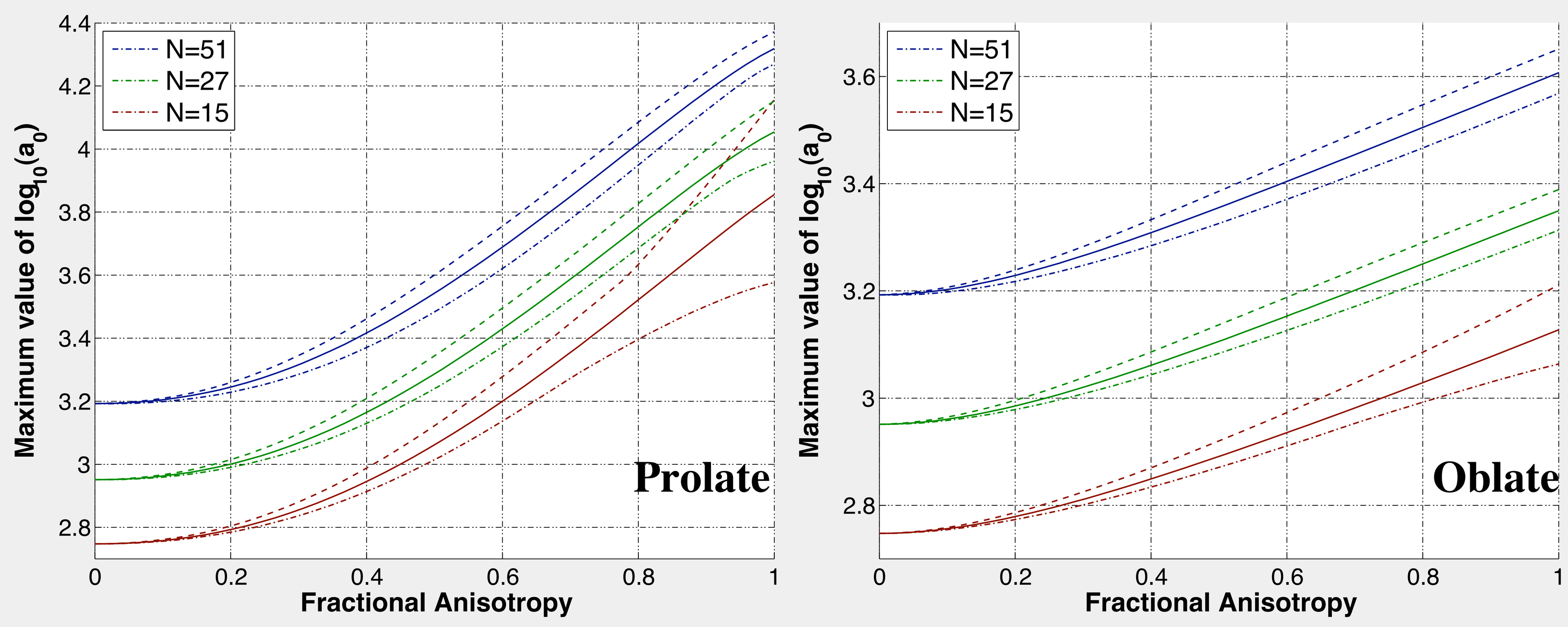
Results and discussion

Simplified scenario with $a_i = a$

$$\mathrm{MSE} = \left(\tfrac{a^{-1}}{2} - \tfrac{a^{-2}}{4}(3L-4)\right)\mathrm{tr}\left(\left(G^TG\right)^{-1}\right) + a^{-2}\tfrac{(L-1)^2N^2}{4}v^T\left(G^TG\right)^{-2}v$$
$$\simeq \tfrac{29.3}{N}\left(\tfrac{a^{-1}}{2} - \tfrac{a^{-2}}{4}(3L-4)\right) + a^{-2}\tfrac{3(L-1)^2}{4} \text{ (empirical approximation)}$$
$$= \tfrac{29.3}{N \cdot \mathrm{NEX}}\left(\tfrac{a^{-1}}{2} - \tfrac{a^{-2}}{4\,\mathrm{NEX}}(3L-4)\right) + a^{-2}\tfrac{3(L-1)^2}{4\,\mathrm{NEX}^2} \text{ (for multiple repetitions)}$$



Left: bias and variance in the DWI signals as a function of a for different numbers of coils; we represent true and empirically computed values together with our approximations. ***Right***: minimum number of receiving coils required (for each a and N) so that the (squared) bias equals the variance in the DT components



Log–plot of the maximum value of $a_0 = LA_0^2/2\sigma^2$ which makes the squared bias equal to the variance in the components of the DT for different tensor shapes. We show minimum, mean and maximum values among all possible tensor orientations. Typical values are used: $L = 8$, $N = 15$, 27, and 51, $b = 1500s/mm^2$, and Mean Diffusivity (MD) $0.8 \cdot 10^{-3}mm^2/s$.

Conclusions

- The impact of the bias in Rician signals for WLS tensor–fitting is quite small in realistic cases; on the contrary, for non–central χ distributed signals, the bias may be an important source of error, (growing with the number of receiving coils).
- While the variance in the estimation may be reduced increasing the number of gradient directions, this is not the case for the bias. In some cases, increasing the number of gradients will not improve the estimation, since the main source of error will be the bias and not the variance. In these cases, it may be preferable to improve the SNR by increasing NEX.
- The traditional WLS approach is not optimal for non–central χ signals, since the weights commonly used are not those yielding minimum variance; although we have proposed a modification to avoid this problem, it makes necessary to characterize the noise power for all image voxels.

References

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